

DUALITY ON FOCK SPACES AND COMBINATORIAL ENERGY FUNCTIONS

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ABSTRACT. We generalize in a combinatorial way the notion of the energy function of affine type A on a sequence of row or column tableaux to the case of a more general class of modules over a general linear Lie superalgebra \mathfrak{g} based on a Howe duality of type $(\mathfrak{g}, \mathfrak{gl}_n)$ on various Fock spaces.

1. INTRODUCTION

The Kostka-Foulkes polynomials are natural q -deformation of Kostka numbers. They appear as the entries of a transition matrix between Schur functions and Hall-Littlewood functions, and also coincide with the Lusztig's q -weight multiplicities of type A (cf. [29]). One of the most important and interesting properties of Kostka-Foulkes polynomials is that they have non-negative integral coefficients. In [27] Lascoux and Schützenberger introduced the notion of charge statistic on semistandard Young tableaux, and proved this positivity of Kostka-Foulkes polynomials.

In [30] Nakayashiki-Yamada showed that the energy function on a finite affine crystal associated to a tensor product of symmetric (or exterior) powers of the natural representation of \mathfrak{gl}_ℓ is given by the Lascoux and Schützenberger's charge. This also gives another combinatorial realization of the Kostka-Foulkes polynomials as q -deformed decomposition multiplicities. Indeed, if we understand the energy function on a sequence of row (or column) tableaux as a statistic on the corresponding non-negative integral (or binary) matrix, then the Nakayashiki-Yamada's result implies that the energy of a given matrix is equal to the charge of its associated recording tableau under the RSK correspondence. We also refer the reader to for example, [22, 31, 32] and references therein for a generalization of the work [30] to the case of a tensor product of arbitrary Kirillov-Reshetikhin crystals of affine type A .

The purpose of this paper is to understand and generalize the energy function of type $A_{\ell-1}^{(1)}$ on a sequence of row (or column) tableaux in another direction from a

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viewpoint of duality principle due to Howe [13]. Let $(\mathcal{A}, \mathcal{B})$ be a pair of countable \mathbb{Z}_2 -graded totally ordered sets. We consider a Lie superalgebra \mathfrak{g} of type A and a semisimple \mathfrak{g} -module \mathcal{F} associated to $(\mathcal{A}, \mathcal{B})$. Following [13] (see also [5]), one can show that \mathfrak{g} forms a dual pair with \mathfrak{gl}_n on $\mathcal{F}^{\otimes n}$ for $n \geq 1$, giving a family of irreducible \mathfrak{g} -modules $L_{\mathfrak{g}}(\lambda)$ appearing in the $(\mathfrak{g}, \mathfrak{gl}_n)$ -decomposition

$$(1.1) \quad \mathcal{F}^{\otimes n} \cong \bigoplus_{\lambda \in H_{\mathfrak{g},n}} L_{\mathfrak{g}}(\lambda) \otimes L_n(\lambda),$$

where $H_{\mathfrak{g},n}$ is a subset of \mathbb{Z}_+^n , the set of generalized partitions of length n , and $L_n(\lambda)$ is the finite-dimensional irreducible \mathfrak{gl}_n -module corresponding to λ (Theorem 2.5).

The decomposition (1.1) is the classical $(\mathfrak{gl}_{\ell}, \mathfrak{gl}_n)$ -duality with $L_{\mathfrak{g}}(\lambda)$ an irreducible polynomial \mathfrak{gl}_{ℓ} -module, when \mathcal{A} is finite with ℓ even elements and $\mathcal{B} = \emptyset$. Moreover, under suitable choices of $(\mathcal{A}, \mathcal{B})$, $\{L_{\mathfrak{g}}(\lambda) \mid n \geq 1, \lambda \in H_{\mathfrak{g},n}\}$ may also include various interesting families of irreducible modules, which forms a semisimple tensor category, for example, the integrable highest weight modules over \mathfrak{gl}_{∞} , infinite-dimensional unitarizable modules over \mathfrak{gl}_{p+q} called a holomorphic discrete series, the irreducible polynomial modules over a general linear Lie superalgebra $\mathfrak{gl}_{p|q}$, and so on (cf. [2, 3, 4, 5, 8, 13, 16, 21]). A uniform combinatorial character formula for $L_{\mathfrak{g}}(\lambda)$ was given by the first author [24] in terms of certain pairs of Young tableaux, which we call *parabolically semistandard tableaux of shape λ (of level n)*.

Now, we consider a \mathfrak{g} -module

$$(1.2) \quad V_{\mathfrak{g}}(\mu) = L_{\mathfrak{g}}(\mu_1) \otimes \cdots \otimes L_{\mathfrak{g}}(\mu_n) \subset \mathcal{F}^{\otimes n},$$

for $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n$ with $\mu_1, \dots, \mu_n \in H_{\mathfrak{g},1}$, which is semisimple and decomposes into $L_{\mathfrak{g}}(\lambda)$'s for $\lambda \in H_{\mathfrak{g},n}$ with finite multiplicity given by the classical Kostka number $K_{\lambda\mu}$ [24]. Then one can define a q -deformed character of $V_{\mathfrak{g}}(\mu)$ by replacing the multiplicity $K_{\lambda\mu}$ of $L_{\mathfrak{g}}(\lambda)$ in $V_{\mathfrak{g}}(\mu)$ with the corresponding Kostka-Foulkes polynomial $K_{\lambda\mu}(q)$, which can be viewed as a natural \mathfrak{g} -analogue of modified Hall-Littlewood function.

As a main result, we introduce a purely combinatorial statistic called a *combinatorial energy function* on the n -tuples of parabolically semistandard tableaux of level 1 associated to (1.2), which generalizes the usual energy function of type $A_{\ell-1}^{(1)}$ on a sequence of row (or column) tableaux, and also produces the q -deformed character of $V_{\mathfrak{g}}(\mu)$ in a bijective way (Theorem 4.5). The main ingredient of our proof is an analogue of RSK algorithm for the decomposition (1.1) of $\mathcal{F}^{\otimes n}$ as a $(\mathfrak{g}, \mathfrak{gl}_n)$ -module [24], which is proved here to be an isomorphism of \mathfrak{gl}_n -crystals (Theorem 4.8). Another important one is an intrinsic characterization of the charge statistic on regular \mathfrak{gl}_n -crystals, which is deduced by combining the result in [30] and a bicrystal structure on the classical RSK correspondence (Theorem 3.2).

We remark that as in the case of the classical $(\mathfrak{gl}_\ell, \mathfrak{gl}_n)$ -duality it would be very interesting to find a representation theoretic meaning of our combinatorial energy function in terms of representations of a quantum (super)algebra associated to an affinization of \mathfrak{g} , especially when $\mathfrak{g} = \mathfrak{gl}_\infty$ with $L_{\mathfrak{g}}(\lambda)$ the integrable highest weight module, or $\mathfrak{g} = \mathfrak{gl}_{p|q}$ with $L_{\mathfrak{g}}(\lambda)$ the irreducible polynomial module [11, 34].

The paper is organized as follows. In Section 2, we recall the notion of parabolically semistandard tableaux and related results. In Section 3, we review the energy function of affine type $A_{\ell-1}^{(1)}$, and the charge statistic on regular \mathfrak{gl}_n -crystals together with its new intrinsic characterization. In Section 4, we introduce a combinatorial energy function, and then show that the associated q -deformed decomposition multiplicities recover the usual Kostka-Foulkes polynomial.

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2. PARABOLICALLY SEMISTANDARD TABLEAUX

2.1. Semistandard tableaux. Let us briefly recall necessary background on semistandard tableaux (cf. [7, 24]). Let \mathcal{P} be the set of partitions, where we often identify a partition with its Young diagram as usual. Throughout this paper, \mathcal{A} (or \mathcal{B}) denotes a countable \mathbb{Z}_2 -graded set (that is, $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$) with a total order $<$. By convention, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{<0}$ denote the set of positive, non-negative, and negative integers with the usual total order and even degree, respectively. For each $\mathbb{Z}_{>0}$, we put $[n] = \{1 < 2 < \cdots < n\}$ and $[-n] = \{-n < \cdots < -2 < -1\}$ with even degree. We assume that $\mathcal{A}' = \{a' \mid a \in \mathcal{A}\}$ is the set with the total order $a'_1 < a'_2$ for $a_1 < a_2 \in \mathcal{A}$ and the opposite \mathbb{Z}_2 -grading.

For a skew Young diagram λ/μ , an \mathcal{A} -semistandard tableau T of shape λ/μ is a filling of λ/μ with entries in \mathcal{A} such that (1) the entries in rows and columns are weakly increasing from left to right and from top to bottom, respectively, (2) the entries in \mathcal{A}_0 are strictly increasing in each column, (3) the entries in \mathcal{A}_1 are strictly increasing in each row. Let $\text{sh}(T)$ denote the shape of T , and $w_{\text{col}}(T)$ (resp. $w_{\text{row}}(T)$) the word with letters in \mathcal{A} obtained by reading the entries column by column (resp. row by row) from right to left (resp. from bottom to top), and in each column (resp. row) from top to bottom (resp. from left to right). Let $SST_{\mathcal{A}}(\lambda/\mu)$ be the set of all \mathcal{A} -semistandard tableaux of shape λ/μ . We set $\mathcal{P}_{\mathcal{A}} = \{\lambda \in \mathcal{P} \mid SST_{\mathcal{A}}(\lambda) \neq \emptyset\}$. For example, $\mathcal{P}_n := \mathcal{P}_{[n]} = \{\lambda \in \mathcal{P} \mid \ell(\lambda) \leq n\}$, where $\ell(\lambda)$ is the length of λ .

Let $P_{\mathcal{A}} = \bigoplus_{a \in \mathcal{A}} \mathbb{Z}\epsilon_a$ be the free abelian group with the basis $\{\epsilon_a \mid a \in \mathcal{A}\}$, and let $\mathbf{x}_{\mathcal{A}} = \{x_a \mid a \in \mathcal{A}\}$ be the set of commuting formal variables indexed by \mathcal{A} . For $T \in SST_{\mathcal{A}}(\lambda/\mu)$, let $\text{wt}_{\mathcal{A}}(T) = \sum_{a \in \mathcal{A}} m_a \epsilon_a \in P_{\mathcal{A}}$ be the weight of T , where m_a is the

number of occurrences of a in T , and put $\mathbf{x}_A^T = \prod_{a \in A} x_a^{m_a}$. We define the character of $SST_A(\lambda/\mu)$ to be $s_{\lambda/\mu}(\mathbf{x}_A) = \sum_{T \in SST_A(\lambda/\mu)} \mathbf{x}_A^T$.

For $a \in A$ and $T \in SST_A(\lambda)$, $(T \leftarrow a)$ denotes the tableau obtained by the Schensted's column bumping algorithm, and $(a \rightarrow T)$ the tableau obtained by the row bumping algorithm. For $T \in SST_A(\lambda)$ and $T' \in SST_A(\mu)$, we set $(T \leftarrow T') = (((T \leftarrow c_1) \cdots) \leftarrow c_t)$ and $(T' \rightarrow T) = (r_t \rightarrow (\cdots (r_1 \rightarrow T)))$, where $w_{\text{col}}(T') = c_1 \cdots c_t$ and $w_{\text{row}}(T') = r_1 \cdots r_t$.

Let $\mu = (\mu_1, \dots, \mu_r)$ be a sequence of non-negative integers. For $(T_1, \dots, T_r) \in SST_A(\mu_1) \times \cdots \times SST_A(\mu_r)$, let $S_k = (((T_1 \leftarrow T_2) \cdots) \leftarrow T_k)$ for $1 \leq k \leq r$. We define $\varrho_{\text{col}}(T_1, \dots, T_r) = (S, S_R)$, where $S = S_r$ and S_R is the $[r]$ -semistandard tableau of shape $\text{sh}(S)$ obtained by filling $\text{sh}(S_k)/\text{sh}(S_{k-1})$ with k for $1 \leq k \leq r$. Similarly, we define $\varrho_{\text{row}}(T_1, \dots, T_r) = (S', S'_R)$, where $S' = (T_r \rightarrow (\cdots (T_2 \rightarrow T_1)))$. Then we have bijections

$$(2.1) \quad \varrho_{\text{col}}, \varrho_{\text{row}} : SST_A(\mu_1) \times \cdots \times SST_A(\mu_r) \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_A} SST_A(\lambda) \times SST_{[r]}(\lambda)_\mu,$$

where $SST_{[r]}(\lambda)_\mu = \{T \in SST_{[r]}(\lambda) \mid \text{wt}_{[r]}(T) = \sum_{i=1}^r \mu_i \epsilon_i\}$.

Let \mathcal{A}^π be \mathcal{A} as a \mathbb{Z}_2 -graded set with the reverse total order of \mathcal{A} . For $T \in SST_A(\lambda/\mu)$, define T^π to be the tableau obtained after 180°-rotation of T , which is an \mathcal{A}^π -semistandard tableau. Let $\mathcal{A} * \mathcal{B}$ be the \mathbb{Z}_2 -graded set $\mathcal{A} \sqcup \mathcal{B}$ with the extended total order defined by $x < y$ for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$. For $S \in SST_A(\mu)$ and $T \in SST_B(\lambda/\mu)$, define $S * T$ to be the tableau of shape λ given by gluing S and T so that $S * T \in SST_{A*B}(\lambda)$.

Example 2.1. Note that the ordered sets $\mathbb{Z}'_{>0}$ and $\mathbb{Z}'_{\geq 0}$ have only elements of odd degree. Letting $P = \begin{array}{|c|c|c|c|} \hline 1' & 2' & 3' & 4' \\ \hline 1' & 2' & 4' & \\ \hline \end{array} \in SST_{\mathbb{Z}'_{>0}}(4, 3)$, we have

$$P^\pi = \begin{array}{|c|c|c|c|} \hline 4' & 2' & 1' & \\ \hline 4' & 3' & 2' & 1' \\ \hline \end{array} \in SST_{(\mathbb{Z}'_{>0})^\pi}((4, 4)/(1)).$$

If we consider

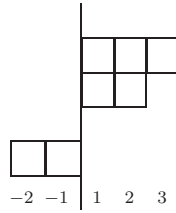
$$H = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & & & \\ \hline \end{array} \in SST_{[3]}(4, 1),$$

$$T = \begin{array}{cccccccc} & & & & 2' & 4' & 5' & 6' & 7' \\ & & & & \hline & 0' & 3' & 4' & 5' & & & & \\ & \hline 0' & 1' & & & & & & & \end{array} \in SST_{\mathbb{Z}'_{\geq 0}}((9, 5, 2)/(4, 1)),$$

then we have

$$H * T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2' & 4' & 5' & 6' & 7' \\ \hline 2 & 0' & 3' & 4' & 5' & & & & \\ \hline 0' & 1' & & & & & & & \\ \hline \end{array} \in SST_{[3]*\mathbb{Z}'_{\geq 0}}(9, 5, 2).$$

2.2. Rational semistandard tableaux. Let us recall the notion of rational semistandard tableaux [33]. Let $\mathbb{Z}_+^n = \{(\lambda_1, \dots, \lambda_n) \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \dots \geq \lambda_n\}$ be the set of generalized partitions of length n . We may identify $\lambda \in \mathbb{Z}_+^n$ with a generalized Young diagram. For example, $\lambda = (3, 2, 0, -2) \in \mathbb{Z}_+^4$ corresponds to



where the non-zero integers indicate the column indices.

For $\lambda \in \mathbb{Z}_+^n$, a *rational semistandard tableau* T of shape λ is a filling of λ with entries in $[n] \sqcup [-n]$ such that (1) the subtableau with columns of positive indices is $[n]$ -semistandard, (2) the subtableau with columns of negative indices is $[-n]$ -semistandard, (3) if $b_1 < \dots < b_s$ (resp. $-b'_1 < \dots < -b'_t$) are the entries in the 1st (resp. -1 st) column with $s + t \leq n$, then $b''_i \leq b_i$ for $1 \leq i \leq s$, where $\{b''_1 < \dots < b''_{n-t}\} = [n] \setminus \{b'_1, \dots, b'_t\}$. Let us call n the *rank* of T , and define the weight of T to be $\text{wt}_{[n]}(T) = \sum_{i \in [n]} (m_i^+ - m_i^-) \epsilon_i$, where m_i^\pm is the number of occurrences of $\pm i$ in T . We also use the same notation $SST_{[n]}(\lambda)$ to denote the set of rational semistandard tableaux of shape λ . For example,

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline \end{array} \begin{array}{|c|c|} \hline -4 & -3 \\ \hline \end{array} \in SST_{[4]}(3, 2, 0, -2)$$

with $\text{wt}_{[4]}(T) = 2\epsilon_1 + 2\epsilon_2 - \epsilon_4$.

For $0 \leq t \leq n$, let T be a tableau in $SST_{[n]}(0^{n-t}, (-1)^t)$ with the entries $-b_1, \dots, -b_t$. We denote by $\sigma(T)$ the tableau in $SST_{[n]}(1^{n-t}, 0^t)$ with the entries $[n] \setminus \{b_1, \dots, b_t\}$. For an arbitrary tableau $T \in SST_{[n]}(\lambda)$, by applying σ to the -1 st column of T , we have a bijection

$$(2.2) \quad \sigma : SST_{[n]}(\lambda) \rightarrow SST_{[n]}(\lambda + (1^n)),$$

where $\text{wt}_{[n]}(\sigma(T)) = \text{wt}_{[n]}(T) + \sum_{i=1}^n \epsilon_i$. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_n$ and $T \in SST_{[n]}(\lambda)$. For $d \geq \lambda_1$, we set $\delta_d(\lambda) = (d^n) - (\lambda_n, \dots, \lambda_1)$ and $\delta_d(T) = (\sigma^{-d}(T))^\pi \in$

$SST_{[-n]^\pi}(\delta_d(\lambda))$. Identifying $-k \in [-n]^\pi$ with $k \in [n]$, we get a bijection

$$(2.3) \quad \delta_d : SST_{[n]}(\lambda) \rightarrow SST_{[n]}(\delta_d(\lambda)).$$

Example 2.2. Let $n = 3$ and

$$Q = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & \\ \hline \end{array} \in SST_{[3]}(4, 3, 0).$$

Then we have $\delta^{-4}(4, 3, 0) = (4, 1, 0)$ and $\sigma^{-4}(Q) = \begin{array}{|c|c|c|c|} \hline & & & -3 \\ \hline -3 & -2 & -1 & -1 \\ \hline \end{array}$. Thus

$$\delta_4(Q) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 3 & & & \\ \hline \end{array} \in SST_{[3]}(4, 1, 0).$$

2.3. Parabolically semistandard tableaux. Now, we review the notion of parabolically semistandard tableaux¹ introduced in [24] to study a combinatorial aspect of Howe dual pairs of type A .

Let $\lambda \in \mathbb{Z}_+^n$ be given. A *parabolically semistandard tableau of shape λ with respect to $(\mathcal{A}, \mathcal{B})$* is a pair of tableaux (T^+, T^-) such that

$$T^+ \in SST_{\mathcal{A}}((\lambda + (d^n))/\mu), \quad T^- \in SST_{\mathcal{B}}((d^n)/\mu),$$

for some integer $d \geq 0$ and $\mu \in \mathcal{P}_n$ satisfying (1) $\lambda + (d^n) \in \mathcal{P}_n$, (2) $\mu \subset (d^n)$, $\mu \subset \lambda + (d^n)$. We call n the *level* of T and define the weight of T to be

$$\text{wt}_{\mathcal{A}/\mathcal{B}}(T) = \text{wt}_{\mathcal{A}}(T^+) - \text{wt}_{\mathcal{B}}(T^-) \in P_{\mathcal{A}} \oplus P_{\mathcal{B}}.$$

We denote by $SST_{\mathcal{A}/\mathcal{B}}(\lambda)$ the set of parabolically semistandard tableaux of shape λ .

Roughly speaking, $T \in SST_{\mathcal{A}/\mathcal{B}}(\lambda)$ is a pair of an \mathcal{A} -semistandard tableau T^+ and a \mathcal{B} -semistandard tableau T^- , the difference of whose shapes is λ . For example, if $\mathcal{A} = \mathcal{B} = \mathbb{Z}_{>0}$, then the pair (T^+, T^-) with

$$T^+ = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & 4 & \\ \hline 2 & 3 & 3 & & & \\ \hline 4 & \cdot & \cdot & & & \\ \hline \end{array} \quad T^- = \begin{array}{|c|c|c|} \hline & 1 & \cdot & \cdot & \cdot \\ \hline & 1 & 2 & \cdot & \cdot \\ \hline 2 & 2 & 4 & & \\ \hline 3 & 3 & 5 & & \\ \hline \end{array}$$

belongs to $SST_{\mathcal{A}/\mathcal{B}}((3, 2, 0, -2))$, where the vertical lines in T^+ and T^- correspond to the one in the generalized partition $\lambda = (3, 2, 0, -2)$, and the bold-faced entries denote ones in the overlapping parts of $\text{sh}(T^+)$ and $\text{sh}(T^-)$. In this case, we have $\text{sh}(T^+) = \lambda + (3^4)/(2, 1, 0, 0)$, and $\text{sh}(T^-) = (3^4)/(2, 1, 0, 0)$.

¹These were called \mathcal{A}/\mathcal{B} -semistandard tableaux in [24].

Let us describe an analogue of RSK correspondence for parabolically semistandard tableaux [24]. From now on, we assume that \mathcal{A} and \mathcal{B} are disjoint sets. Let

$$\mathcal{F}_{\mathcal{A}/\mathcal{B}} = \bigsqcup_{k \in \mathbb{Z}} SST_{\mathcal{A}/\mathcal{B}}(k)$$

be the set of all parabolically semistandard tableaux of level 1, and $\mathcal{F}_{\mathcal{A}/\mathcal{B}}^n$ its n -fold product. Let $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{F}_{\mathcal{A}/\mathcal{B}}^n$ be given with $T_i = (T_i^+, T_i^-)$. We associate a pair $(P_{\mathbf{T}}, Q_{\mathbf{T}})$, where $P_{\mathbf{T}}$ is a parabolically semistandard tableau of level n and $Q_{\mathbf{T}}$ is a rational semistandard tableau of rank n determined by the following steps:

(κ -1) Let

$$(P, Q) = \varrho_{\text{col}}((T_1^-)^\pi, \dots, (T_n^-)^\pi).$$

Put $T^- = P^\pi$ and write $\text{sh}(T^-) = (d^n)/\mu$ for some $d \geq 0$ and $\mu \in \mathcal{P}_n$.

(κ -2) Let $Q^\vee = \delta_d(Q)$, which is of shape μ , and let $\nu = (\nu_1, \dots, \nu_n)$, where $\text{wt}_{[n]}(Q^\vee) = \sum_{i \in [n]} \nu_i \epsilon_i$. By (2.1), there exist unique $S_i \in SST_{[n]}(\nu_i)$ for $1 \leq i \leq n$ such that

$$\varrho_{\text{row}}(S_1, \dots, S_n) = (H^\mu, Q^\vee) \in SST_{[n]}(\mu) \times SST_{[n]}(\mu)_\nu,$$

where H^μ is the tableau of shape μ with weight $\sum_{i \in [n]} \mu_i \epsilon_i$.

(κ -3) For $1 \leq i \leq n$, put $U_i = S_i * T_i^+$, which is an $[n] * \mathcal{A}$ -semistandard tableau. Using (2.1) once again, we let

$$(U, U_R) = \varrho_{\text{row}}(U_1, \dots, U_n) \in SST_{[n]*\mathcal{A}}(\lambda + (d)^n) \times SST_{[n]}(\lambda + (d)^n),$$

for some $\lambda \in \mathbb{Z}_+^n$.

(κ -4) Since $i < a$ for all $i \in [n]$ and $a \in \mathcal{A}$ in $[n] * \mathcal{A}$, we have $U = H^\mu * T^+$ for some $T^+ \in SST_{\mathcal{A}}(\lambda + (d)^n/\mu)$. Finally, we define

$$P_{\mathbf{T}} = (T^+, T^-) \in SST_{\mathcal{A}/\mathcal{B}}(\lambda),$$

$$Q_{\mathbf{T}} = \sigma^{-d}(U_R) \in SST_{[n]}(\lambda).$$

Example 2.3. Let $\mathcal{A} = \mathbb{Z}'_{\geq 0}$ and $\mathcal{B} = \mathbb{Z}'_{< 0}$. Note that \mathcal{A} and \mathcal{B} have only elements of odd degree. Consider

$$\mathbf{T} = (T_1, T_2, T_3) \in SST_{\mathcal{A}/\mathcal{B}}(3) \times SST_{\mathcal{A}/\mathcal{B}}(1) \times SST_{\mathcal{A}/\mathcal{B}}(0) \subset \mathcal{F}_{\mathcal{A}/\mathcal{B}}^3,$$

where

$$T_1 = (T_1^+, T_1^-) = (\boxed{0'} \boxed{1'} \boxed{3'} \boxed{4'} \boxed{5'} , \boxed{-4'} \boxed{-3'}),$$

$$T_2 = (T_2^+, T_2^-) = (\boxed{0'} \boxed{2'} \boxed{6'} \boxed{7'} , \boxed{-4'} \boxed{-2'} \boxed{-1'}),$$

$$T_3 = (T_3^+, T_3^-) = (\boxed{4'} \boxed{5'} , \boxed{-2'} \boxed{-1'}).$$

Then, we have

$$\begin{aligned} \varrho_{\text{col}}((T_1^-)^\pi, (T_2^-)^\pi, (T_3^-)^\pi) &= \varrho_{\text{col}}(\boxed{-3' -4'}, \boxed{-1' -2' -4'}, \boxed{-1' -2'}) \\ &= (P, Q) = \left(\begin{array}{|c|c|c|c|} \hline -1' & -2' & -3' & -4' \\ \hline -1' & -2' & -4' & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & \\ \hline \end{array} \right) \in SST_{(\mathbb{Z}'_{<0})^\pi}(4, 3, 0) \times SST_{[3]}(4, 3, 0), \end{aligned}$$

which yield $T^- = \begin{array}{|c|c|c|c|} \hline & -4' & -2' & -1' \\ \hline -4' & -3' & -2' & -1' \\ \hline \end{array}$ (see Example 2.1). We choose $d = 4$ and $\mu = (4, 1, 0)$. So we have

$$Q^\vee = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 3 & & & \\ \hline \end{array} \in SST_{[3]}(\mu)$$

with $\text{wt}_{[3]}(Q^\vee) = 2\epsilon_1 + \epsilon_2 + 2\epsilon_3$, which implies $\nu = (2, 1, 2)$ (see Example 2.2). It follows from

$$\varrho_{\text{row}}(\boxed{1 \ 1}, \boxed{2}, \boxed{1 \ 1}) = \left(\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 3 & & & \\ \hline \end{array} \right) \in SST_{[3]}(\mu) \times SST_{[3]}(\mu)_\nu$$

that

$$\begin{aligned} U_1 &= \boxed{1 \ 1 \ 0' \ 1' \ 3' \ 4' \ 5'}, \\ U_2 &= \boxed{2 \ 0' \ 2' \ 6' \ 7'}, \\ U_3 &= \boxed{1 \ 1 \ 4' \ 5'}. \end{aligned}$$

Thus, we have $\varrho_{\text{row}}(U_1, U_2, U_3) = (U, U_R)$, where

$$U = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2' & 4' & 5' & 6' & 7' \\ \hline 2 & 0' & 3' & 4' & 5' & & & & \\ \hline 0' & 1' & & & & & & & \\ \hline \end{array}, \quad U_R = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 2 & 3 & 3 & & & & \\ \hline 3 & 3 & & & & & & & \\ \hline \end{array}.$$

Since $d = 4$ and $U = H^\mu * T^+$ (see Example 2.1), we have $\lambda = (5, 1, -2)$ and

$$\begin{aligned} P_{\mathbf{T}} &= \left(\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & & & & 2' & 4' & 5' & 6' & 7' \\ \hline & 0' & 3' & 4' & 5' & & & & \\ \hline 0' & 1' & & & & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline -4' & -2' & -1' \\ \hline -4' & -3' & -2' & -1' \\ \hline \end{array} \right), \\ Q_{\mathbf{T}} &= \begin{array}{|c|c|c|c|c|} \hline & 1 & 1 & 1 & 2 & 2 \\ \hline & 3 & & & & \\ \hline -3 & -2 & & & & \\ \hline \end{array}. \end{aligned}$$

Theorem 2.4. [24, Theorem 4.1] *The map $\mathbf{T} \mapsto (P_{\mathbf{T}}, Q_{\mathbf{T}})$ gives a bijection*

$$\kappa_{\mathcal{A}/\mathcal{B}} : \mathcal{F}_{\mathcal{A}/\mathcal{B}}^n \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}} SST_{\mathcal{A}/\mathcal{B}}(\lambda) \times SST_{[n]}(\lambda),$$

where $\mathcal{P}_{\mathcal{A}/\mathcal{B},n} = \{\lambda \in \mathbb{Z}_+^n \mid SST_{\mathcal{A}/\mathcal{B}}(\lambda) \neq \emptyset\}$.

For $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{F}_{\mathcal{A}/\mathcal{B}}^n$, we put

$$\text{wt}_{\mathcal{A}/\mathcal{B}}(\mathbf{T}) = \sum_{i=1}^n \text{wt}_{\mathcal{A}/\mathcal{B}}(T_i) \in P_{\mathcal{A}} \oplus P_{\mathcal{B}}, \quad \text{wt}_{[n]}(\mathbf{T}) = \sum_{i=1}^n m_i \epsilon_i \in P_{[n]},$$

where $T_i \in SST_{\mathcal{A}/\mathcal{B}}(m_i)$ for $1 \leq i \leq n$. One can observe that

$$\text{wt}_{\mathcal{A}/\mathcal{B}}(\mathbf{T}) = \text{wt}_{\mathcal{A}/\mathcal{B}}(P_{\mathbf{T}}), \quad \text{wt}_{[n]}(\mathbf{T}) = \text{wt}_{[n]}(Q_{\mathbf{T}}),$$

and hence $\kappa_{\mathcal{A}/\mathcal{B}}$ preserves the weights.

We define the character of $SST_{\mathcal{A}/\mathcal{B}}(\lambda)$ to be

$$(2.4) \quad S_{\lambda}^{\mathcal{A}/\mathcal{B}} = \sum_{T \in SST_{\mathcal{A}/\mathcal{B}}(\lambda)} \mathbf{x}_{\mathcal{A}/\mathcal{B}}^T,$$

where $\mathbf{x}_{\mathcal{A}/\mathcal{B}}^T = \mathbf{x}_{\mathcal{A}}^{T^+} (\mathbf{x}_{\mathcal{B}}^{T^-})^{-1}$ for $T = (T^+, T^-) \in SST_{\mathcal{A}/\mathcal{B}}(\lambda)$. Then Theorem 2.4 establishes the following Cauchy-type identity:

$$(2.5) \quad \prod_{i \in [n]} \frac{\prod_{a \in \mathcal{A}_1} (1 + x_a x_i) \prod_{b \in \mathcal{B}_1} (1 + x_b^{-1} x_i^{-1})}{\prod_{a \in \mathcal{A}_0} (1 - x_a x_i) \prod_{b \in \mathcal{B}_0} (1 - x_b^{-1} x_i^{-1})} = \sum_{\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}} S_{\lambda}^{\mathcal{A}/\mathcal{B}} s_{\lambda}(\mathbf{x}_{[n]}).$$

Here $s_{\lambda}(\mathbf{x}_{[n]})$ is the Laurent Schur polynomial corresponding to $\lambda \in \mathbb{Z}_+^n$.

Note that, when $\mathcal{B} = \emptyset$, we have $\mathcal{P}_{\mathcal{A}/\mathcal{B},n} = \mathcal{P}_{\mathcal{A}} \cap \mathcal{P}_n$, and $S_{\lambda}^{\mathcal{A}/\mathcal{B}} = s_{\lambda}(\mathbf{x}_{\mathcal{A}})$, which is the usual (super) Schur function or polynomial corresponding to λ , and the identity (2.5) recovers the well-known Cauchy identity. So a non-trivial generalization of Schur functions or more interesting cases occur when both \mathcal{A} and \mathcal{B} are non-empty.

2.4. Howe duality and irreducible characters. The notion of parabolically semistandard tableaux and its RSK with rational semistandard tableaux for the general linear Lie algebra \mathfrak{gl}_n gives a unified combinatorial interpretation of various dualities of $(\mathfrak{g}, \mathfrak{gl}_n)$, where \mathfrak{g} is a general linear Lie superalgebra associated to $(\mathcal{A}, \mathcal{B})$. We assume that the base field is \mathbb{C} .

Let us explain it in more detail. For an arbitrary countable \mathbb{Z}_2 -graded totally ordered set S , let V_S be a superspace with basis $\{v_s \mid s \in S\}$, and let \mathfrak{gl}_S be the general linear Lie superalgebra spanned by the elementary matrices $E_{ss'}$ for $s, s' \in S$, where the parity of $E_{ss'}$ is given by the sum of the parities of s and s' (cf. [15]).

Now we consider $\mathfrak{g} = \mathfrak{gl}_{\mathcal{C}}$ with $\mathcal{C} = \mathcal{B} * \mathcal{A}$. Let

$$\mathcal{F} = S(V_{\mathcal{A}} \oplus V_{\mathcal{B}}^{\vee})$$

be the super symmetric algebra generated by $V_A \oplus V_B^\vee$, where V_B^\vee is the restricted dual space of V_B . Recall that \mathcal{F} can be viewed as an irreducible module over a Clifford-Weyl algebra. Following the arguments in [5, Sections 5.1 and 5.4] (cf. [8, 16]), one can define a semisimple action of \mathfrak{g} on \mathcal{F} , and a semisimple action of \mathfrak{gl}_n or GL_n on $\mathcal{F}^{\otimes n}$ for $n \geq 1$ such that $\mathcal{F}^{\otimes n}$ decomposes into a finite-dimensional \mathfrak{gl}_n -modules. Then the actions of \mathfrak{g} and \mathfrak{gl}_n commute with each other, and furthermore the image of \mathfrak{g} in $\text{End}_{\mathbb{C}}(\mathcal{F}^{\otimes n})$ generates $\text{End}_{\mathfrak{gl}_n}(\mathcal{F}^{\otimes n})$. Therefore, we have the following multiplicity-free decomposition as a $(\mathfrak{g}, \mathfrak{gl}_n)$ -module,

$$(2.6) \quad \mathcal{F}^{\otimes n} \cong \bigoplus_{\lambda \in H_{\mathfrak{g},n}} L_{\mathfrak{g}}(\lambda) \otimes L_n(\lambda),$$

for a subset $H_{\mathfrak{g},n}$ of \mathbb{Z}_+^n , where $L_n(\lambda)$ is an irreducible \mathfrak{gl}_n -module with highest weight $\lambda \in H_{\mathfrak{g},n}$, and $L_{\mathfrak{g}}(\lambda)$ is an irreducible \mathfrak{g} -module corresponding to $L_n(\lambda)$. We define the character $\text{ch} L_{\mathfrak{g}}(\lambda)$ to be the trace of the operator $\prod_{c \in \mathcal{C}} x_c^{E_{cc}}$ on $L_{\mathfrak{g}}(\lambda)$ for $\lambda \in H_n$.

Then we have the following decomposition, which is often referred to as Howe duality (for type A) (cf. [2, 3, 4, 8, 13, 16, 21]).

Theorem 2.5. *Let \mathcal{A} and \mathcal{B} be given. For $n \geq 1$, we have*

$$\mathcal{F}^{\otimes n} \cong \bigoplus_{\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}} L_{\mathfrak{g}}(\lambda) \otimes L_n(\lambda),$$

as a $(\mathfrak{g}, \mathfrak{gl}_n)$ -module, that is, $H_{\mathfrak{g},n} = \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$, and the irreducible character $\text{ch} L_{\mathfrak{g}}(\lambda)$ is given by $S_{\lambda}^{\mathcal{A}/\mathcal{B}}$ for $\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$.

Proof. Consider the operator $D = \prod_{c \in \mathcal{C}} x_c^{E_{cc}} \prod_{i \in [n]} x_i^{e_{ii}}$, where e_{ii} is the i -th elementary diagonal matrix in \mathfrak{gl}_n . Taking the trace of D on both sides of (2.6), we have

$$\prod_{i \in [n]} \frac{\prod_{a \in \mathcal{A}_1} (1 + x_a x_i) \prod_{b \in \mathcal{B}_1} (1 + x_b^{-1} x_i^{-1})}{\prod_{a \in \mathcal{A}_0} (1 - x_a x_i) \prod_{b \in \mathcal{B}_0} (1 - x_b^{-1} x_i^{-1})} = \sum_{\lambda \in H_{\mathfrak{g},n}} \text{ch} L_{\mathfrak{g}}(\lambda) s_{\lambda}(\mathbf{x}_{[n]}).$$

Thus by the Cauchy-type identity (2.5) and the linear independence of Laurent Schur polynomials, we conclude that $H_{\mathfrak{g},n} = \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$ and $S_{\lambda}^{\mathcal{A}/\mathcal{B}} = \text{ch} L_{\mathfrak{g}}(\lambda)$ for $\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$. \square

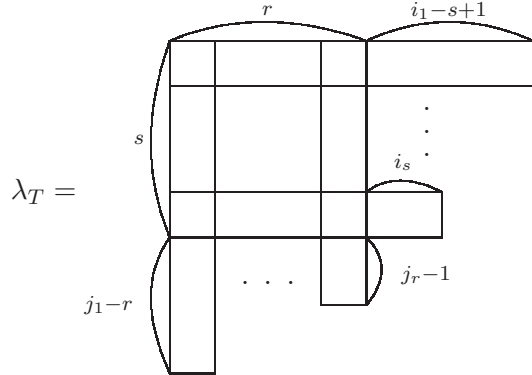
Note that $L_{\mathfrak{g}}(\lambda)$'s are mutually non-isomorphic irreducible \mathfrak{g} -modules for $\lambda \in \bigcup_{n \geq 1} \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$ and the tensor product $L_{\mathfrak{g}}(\mu) \otimes L_{\mathfrak{g}}(\nu)$ for $\mu \in \mathcal{P}_{\mathcal{A}/\mathcal{B},m}$ and $\nu \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$ decomposes into a direct sum of $L_{\mathfrak{g}}(\lambda)$'s for $\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},m+n}$ with finite multiplicity given by a Littlewood-Richardson number (see [24, Theorem 4.7]). Also, $L_{\mathfrak{g}}(\lambda)$ is semisimple over a maximal Levi subalgebra $\mathfrak{l} = \mathfrak{gl}_{\mathcal{A}} \oplus \mathfrak{gl}_{\mathcal{B}}$ of \mathfrak{g} , and expanding $S_{\mathcal{A}/\mathcal{B}}^{\lambda}$ as a linear combination of $s_{\mu}(\mathbf{x}_{\mathcal{A}}) s_{\nu}(\mathbf{x}_{\mathcal{B}}^{-1})$ for $\mu, \nu \in \mathcal{P}$ (see [24, Proposition 3.14]) gives a branching rule with respect to \mathfrak{l} or its associated maximal parabolic sub algebra.

Recall that when \mathcal{A} is finite with $\mathcal{A} = \mathcal{A}_0$ or \mathcal{A}_1 and $\mathcal{B} = \emptyset$, the decomposition in Theorem 2.5 is the classical $(\mathfrak{gl}_\ell, \mathfrak{gl}_n)$ -Howe duality on symmetric algebra or exterior algebra generated by $\mathbb{C}^\ell \otimes \mathbb{C}^n$, where $\ell = |\mathcal{A}|$ (cf. [13]). Below we list some of important examples where both \mathcal{A} and \mathcal{B} are non-empty, and \mathfrak{g} is a usual general linear Lie algebra (see [24] for more detailed exposition).

Example 2.6. (1) If $(\mathcal{A}, \mathcal{B}) = (\mathbb{Z}'_{\geq 0}, \mathbb{Z}'_{< 0})$, then $S_\lambda^{A/B}$ ($\lambda \in \mathbb{Z}_+^n$) is the character of an integrable highest weight module over the general linear Lie algebra \mathfrak{gl}_∞ with highest weight of positive level n . The identity (2.5) corresponds to the $(\mathfrak{gl}_\infty, \mathfrak{gl}_n)$ -duality on the level n fermionic Fock space $\mathcal{F}^{\otimes n}$ [8]. In particular, $SST_{A/B}(k)$ ($k \in \mathbb{Z}$) can be identified with a linear basis of the level 1 fermionic Fock space \mathcal{F} of charge k , which is realized by \mathcal{P} [14, Section 1], by mapping an element

$$T = (\boxed{i'_s} \cdots \boxed{i'_2} \boxed{i'_1} , \boxed{-j'_1} \boxed{-j'_2} \cdots \boxed{-j'_r}),$$

where $i_p \geq 0$, $j_q > 0$ and $s - r = k$, to the following Young diagram



For example, the Young diagrams corresponding to T_1 , T_2 and T_3 in Example 2.3 are $\lambda_{T_1} = (3, 3, 3, 2, 2, 2)$, $\lambda_{T_2} = (7, 7, 4, 3, 1)$ and $\lambda_{T_3} = (6, 6)$, respectively.

(2) If $(\mathcal{A}, \mathcal{B}) = (\mathbb{Z}_{\geq 0}, \mathbb{Z}_{< 0})$, then $S_\lambda^{A/B}$ ($\lambda \in \mathbb{Z}_+^n$) is the character of an irreducible (non-integrable) highest weight module over \mathfrak{gl}_∞ with highest weight of negative level $-n$, which appears in the $(\mathfrak{gl}_\infty, \mathfrak{gl}_n)$ -duality on the level n bosonic Fock space [16].

(3) If $(\mathcal{A}, \mathcal{B}) = ([-q], [p])$ for $p, q \in \mathbb{Z}_{> 0}$, then $S_\lambda^{A/B}$ ($\lambda \in \mathbb{Z}_+^n$) is equal to the character of an infinite-dimensional irreducible \mathfrak{gl}_{p+q} -module, which is unitarizable. The family of irreducible representations appears in the $(\mathfrak{gl}_{p+q}, \mathfrak{gl}_n)$ -duality on the symmetric algebra $S(\mathbb{C}^p \otimes \mathbb{C}^n \oplus \mathbb{C}^{q*} \otimes \mathbb{C}^{n*})$ [21], which are called holomorphic discrete series or oscillator modules.

2.5. Hall-Littlewood functions. Let q be an indeterminate. Fix $n \geq 1$. For $\mu \in \mathcal{P}_n$, let $P_\mu(\mathbf{x}_{[n]}, q)$ be the Hall-Littlewood polynomial in $\mathbf{x}_{[n]}$ associated to μ [29, Chapter III.2]. For $\mu \in \mathbb{Z}_+^n$, we put $P_\mu(\mathbf{x}_{[n]}, q) = (x_1 \cdots x_n)^{-d} P_{\mu+(d^n)}(\mathbf{x}_{[n]}, q)$

for some $d \geq 0$ such that $\mu + (d^n) \in \mathcal{P}_n$, which is independent of d and hence well-defined.

Consider a formal power series $Q_\lambda^{A/B}$ in \mathbf{x}_A and \mathbf{x}_B^{-1} , which is determined by the following Cauchy-type identity:

$$(2.7) \quad \prod_{i \in [n]} \frac{\prod_{a \in A_1} (1 + x_a x_i) \prod_{b \in B_1} (1 + x_b^{-1} x_i^{-1})}{\prod_{a \in A_0} (1 - x_a x_i) \prod_{b \in B_0} (1 - x_b^{-1} x_i^{-1})} = \sum_{\lambda \in \mathbb{Z}_+^n} Q_\lambda^{A/B} P_\lambda(\mathbf{x}_{[n]}, q).$$

The following is a well-known identity:

$$(2.8) \quad s_\lambda(\mathbf{x}_{[n]}) = \sum_{\mu \in \mathbb{Z}_+^n} K_{\lambda\mu}(q) P_\mu(\mathbf{x}_{[n]}, q),$$

for $\lambda \in \mathbb{Z}_+^n$, where $K_{\lambda\mu}(q)$ ($\lambda, \mu \in \mathcal{P}_n$) are the Kostka-Foulkes polynomials or Lusztig's q -weight multiplicities of type A_{n-1} . Here we set $K_{\lambda\mu}(q) = K_{\lambda+(d^n)\mu+(d^n)}(q)$ for $d \geq 1$ with $\lambda + (d^n), \mu + (d^n) \in \mathcal{P}_n$, which is independent of the choice of d . By (2.5), (2.7) and (2.8), we have

$$(2.9) \quad Q_\mu^{A/B} = \sum_{\lambda \in \mathbb{Z}_+^n} K_{\lambda\mu}(q) S_\lambda^{A/B},$$

for $\mu \in \mathbb{Z}_+^n$. Since $K_{\lambda\mu}(q)$ has nonnegative integral coefficients with $K_{\lambda\mu}(1) = |SST_{[n]}(\lambda)_\mu|$, we may view $Q_\mu^{A/B}$ as a q -analogue of the character of

$$(2.10) \quad \mathcal{F}_{A/B}^\mu = SST_{A/B}(\mu_1) \times \cdots \times SST_{A/B}(\mu_n),$$

by Theorem 2.4. Recall that $Q_\mu^{A/B}$ is a modified Hall-Littlewood function for $\mu \in \mathcal{P}$ when $A = \mathbb{Z}_{>0}$ or $\mathbb{Z}'_{>0}$ and $B = \emptyset$, and it can be realized as a graded character of a tensor product of KR crystals with respect to an energy function of affine type A [30].

Our main goal is to introduce a purely combinatorial statistic on $\mathcal{F}_{A/B}^\mu$, which realizes (2.9) as a graded character of $\mathcal{F}_{A/B}^\mu$ for arbitrary A and B , also generalizing the usual energy functions on sequences of row (or column) tableaux.

3. AFFINE CRYSTALS AND CHARGE STATISTIC

3.1. Crystals. Let us give a brief review on crystals (cf. [12, 19]). Let \mathfrak{g} be the Kac-Moody algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$. Let P^\vee be the dual weight lattice, $P = \text{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ the weight lattice, $\Pi^\vee = \{h_i \mid i \in I\}$ the set of simple coroots, and $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots of \mathfrak{g} such that $\langle h_i, \alpha_j \rangle = a_{ij}$ for $i, j \in I$. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of \mathfrak{g} .

A \mathfrak{g} -crystal (or crystal for short) is a set B together with the maps $\text{wt} : B \rightarrow P$, $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \cup \{\mathbf{0}\}$ ($i \in I$) satisfying certain axioms.

For a dominant integral weight Λ for \mathfrak{g} , we denote by $B(\Lambda)$ the crystal associated to the irreducible highest weight $U_q(\mathfrak{g})$ -module with highest weight Λ .

For a crystal B , we denote its dual by B^\vee , which is a set $B^\vee = \{b^\vee \mid b \in B\}$ with

$$\begin{aligned} \text{wt}(b^\vee) &= -\text{wt}(b), \\ \varepsilon_i(b^\vee) &= \varphi_i(b), \quad \varphi_i(b^\vee) = \varepsilon_i(b), \\ \tilde{e}_i(b^\vee) &= \tilde{f}_i(b)^\vee, \quad \tilde{f}_i(b^\vee) = \tilde{e}_i(b)^\vee, \end{aligned}$$

for $b \in B$ and $i \in I$. A tensor product $B_1 \otimes B_2$ of crystals B_1 and B_2 is defined to be a crystal, which is $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, satisfying

$$\begin{aligned} \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \text{wt}(b_1), h_i \rangle\}, \\ \varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_1) + \langle \text{wt}(b_2), h_i \rangle, \varphi_i(b_2)\}, \\ \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \end{aligned}$$

for $i \in I$. Here we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$.

Given $b_1 \in B_1$ and $b_2 \in B_2$, we write $b_1 \equiv b_2$ if there is an isomorphism of crystals $C(b_1) \rightarrow C(b_2)$ mapping b_1 to b_2 , where $C(b_i)$ denotes the connected component of b_i in B_i for $i = 1, 2$.

3.2. A_{n-1} -crystals. Fix a positive integer $n \geq 2$. Suppose that $\mathfrak{g} = A_{n-1}$ or the associated generalized Cartan matrix is of type A_{n-1} with $I = \{1, \dots, n-1\}$. We assume that its weight lattice is $P_n := P_{[n]}$. We often identify $\lambda \in \mathbb{Z}_+^n$ with the dominant integral weight $\sum_{i=1}^n \lambda_i \epsilon_i$. Let $\Delta_{n-1}^+ = \{\epsilon_s - \epsilon_t \mid 1 \leq s < t \leq n\}$ the set of positive roots. The Weyl group is the symmetric group \mathfrak{S}_n on n letters generated by the transposition $r_j = (j \ j+1)$ for $j = 1, \dots, n-1$. From now on, we always denote the associated data of an A_{n-1} -crystal by $\tilde{e}_j, \tilde{f}_j, \varepsilon_j, \varphi_j$ ($j = 1, \dots, n-1$) and wt .

We may regard $[n]$ as $B(\epsilon_1)$ the crystal of the natural representation, and $[-n]$ as its dual. Given $\lambda \in \mathbb{Z}_+^n$, $SST_{[n]}(\lambda)$ has an A_{n-1} -crystal structure by regarding $w_{\text{col}}(T)$ for $T \in SST_{[n]}(\lambda)$ as an element in $[n]^{\otimes p} \otimes [-n]^{\otimes q}$ for some $p, q \geq 0$. Here, we understand that $w_{\text{col}}(T)$ is the word with letters in $[n] \cup [-n]$ obtained from T by column reading as usual. Then we have $SST_{[n]}(\lambda) \cong B(\lambda)$ (cf. [20]). It is not

difficult to see that

$$(3.1) \quad \sigma^d(T) \equiv T, \quad \delta_d(S) \equiv S^\vee,$$

for $T \in SST_{[n]}(\lambda)$ with $\lambda \in \mathbb{Z}_+^n$ and $S \in SST_{[n]}(\mu)$ with $\mu \in \mathcal{P}_n$ up to a shift of weight by $d(\epsilon_1 + \dots + \epsilon_n)$ ($d \in \mathbb{Z}$) or as elements in A_{n-1} -crystals with the weight lattice $P_n/\mathbb{Z}(\epsilon_1 + \dots + \epsilon_n)$ (see (2.2) and (2.3) for σ and δ_d). The crystal equivalence \equiv is also compatible with row and column insertions, that is, $(T \leftarrow a) \equiv T \otimes a$ and $(a \rightarrow T) \equiv a \otimes T$ for $a \in [n]$ and $T \in SST_{[n]}(\lambda)$ with $\lambda \in \mathcal{P}_n$.

3.3. Charge statistic. For $\lambda, \mu \in \mathcal{P}_n$ and $T \in SST_{[n]}(\lambda)_\mu$, we denote by $c(T)$ the *charge* of T , which was introduced by Lascoux and Schützenberger [27]. It is shown in [28] that

$$(3.2) \quad K_{\lambda\mu}(q) = \sum_{T \in SST_{[n]}(\lambda)_\mu} q^{c(T)}.$$

One can naturally induce a charge statistic on a regular A_{n-1} -crystal B as follows: Let $b \in B$ be given. First, note that the connected component $C(b) \subset B$ under \tilde{e}_j and \tilde{f}_j ($j = 1, \dots, n-1$) is isomorphic to $B(\lambda)$ for some $\lambda \in \mathbb{Z}_+^n$. Choose $d \geq 0$ such that $\lambda + (d^n) \in \mathcal{P}_n$. Since $B(\lambda) \cong SST_{[n]}(\lambda + (d^n))$ as a $\{1, \dots, n-1\}$ -colored oriented graph by (3.1), b can be identified with a tableau $T \in SST_{[n]}(\lambda + (d^n))$. Then we define

$$(3.3) \quad \text{charge}(b) = c(T'),$$

where T' is a unique tableau with dominant weight in the \mathfrak{S}_n -orbit of T . By definition of Lascoux and Schützenberger's charge, it is not difficult to see that $\text{charge}(b)$ does not depend on the choice of d . In particular, we define for $\lambda, \mu \in \mathbb{Z}_+^n$

$$(3.4) \quad K_{\lambda\mu}(q) = \sum_{b \in B(\lambda), \text{ wt}(b)=\mu} q^{\text{charge}(b)},$$

which is equal to the usual Kostka-Foulkes polynomial $K_{\lambda+(d^n)\mu+(d^n)}(q)$ for some $d \geq 0$.

Recall that there is an intrinsic characterization of the charge statistic [26], which is described only in terms of the geometry of the crystal graph $B(\lambda)$ for $\lambda \in \mathbb{Z}_+^n$. In Section 3.5, we give another intrinsic characterization, which plays a crucial role in this paper. For this, we need the following statistic on a regular A_{n-1} -crystal B : for $\alpha = \epsilon_s - \epsilon_t \in \Delta_{n-1}^+$ and $b \in B$

$$(3.5) \quad \begin{aligned} \varepsilon_\alpha(b) &= \varepsilon_s(S_{s+1}S_{s+2} \cdots S_{t-1}(b)), \\ \varphi_\alpha(b) &= \varphi_s(S_{s+1}S_{s+2} \cdots S_{t-1}(b)), \end{aligned}$$

where S_j is the \mathfrak{S}_n -action on B associated to r_j . Since $\varphi_\alpha(b) - \varepsilon_\alpha(b) = \langle \text{wt}(b), \alpha^\vee \rangle$, where $\alpha^\vee = h_s + \dots + h_{t-1}$ is the coroot of α , one may think of $\varepsilon_\alpha(b)$ and $\varphi_\alpha(b)$ as

information on an \mathfrak{sl}_2 -string of b with respect to $\alpha = r_{t-1} \dots r_{s+1}(\alpha_s)$. We should remark that they depend on the choice of a simple root conjugate to α . Here we choose it as α_s .

3.4. Affine $A_{\ell-1}^{(1)}$ -crystals and energy function. Fix a positive integer $\ell \geq 2$. Suppose that $\mathfrak{g} = A_{\ell-1}^{(1)}$ with $I = \{0, \dots, \ell-1\}$ and $\mathfrak{g}_0 = A_{\ell-1}$ is the subalgebra of \mathfrak{g} corresponding to $I \setminus \{0\}$. For $1 \leq r \leq \ell-1$, let ϖ_r be the fundamental weight for \mathfrak{g}_0 corresponding to the simple root α_r . For $s \geq 1$, let $B^{r,s}$ denote the *Kirillov-Reshetikhin crystal* (or KR crystal for short) of type $A_{\ell-1}^{(1)}$, which is isomorphic to $B(s\varpi_r)$ as an $A_{\ell-1}$ -crystal [18, 32]. Let $u_{r,s}$ be the unique element in $B^{r,s}$ of weight $s\varpi_r$. For convenience, let us assume that $B^{0,s}$ and $B^{\ell,s}$ are trivial crystals.

Let B_1 and B_2 be two KR crystals with the classical highest weight elements u_1 and u_2 , respectively. Let $\sigma = \sigma_{B_1, B_2} : B_1 \otimes B_2 \longrightarrow B_2 \otimes B_1$ be a unique $A_{\ell-1}^{(1)}$ -crystal isomorphism called the *combinatorial R-matrix*. There exists a function $H = H_{B_1, B_2} : B_1 \otimes B_2 \longrightarrow \mathbb{Z}$ such that H is constant on each connected component in $B_1 \otimes B_2$ as an $A_{\ell-1}$ -crystal and

$$H(\tilde{e}_0(b_1 \otimes b_2)) =$$

$$\begin{cases} H(b_1 \otimes b_2) + 1, & \text{if } \tilde{e}_0(b_1 \otimes b_2) = \tilde{e}_0(b_1) \otimes b_2 \text{ and } \tilde{e}_0(b'_2 \otimes b'_1) = \tilde{e}_0(b'_2) \otimes b'_1, \\ H(b_1 \otimes b_2) - 1, & \text{if } \tilde{e}_0(b_1 \otimes b_2) = b_1 \otimes \tilde{e}_0(b_2) \text{ and } \tilde{e}_0(b'_2 \otimes b'_1) = b'_2 \otimes \tilde{e}_0(b'_1), \\ H(b_1 \otimes b_2), & \text{otherwise,} \end{cases}$$

for $b_1 \otimes b_2 \in B_1 \otimes B_2$ with $b'_2 \otimes b'_1 = \sigma(b_1 \otimes b_2)$. It is well-known that H is unique up to an additive constant and is called the *local energy function* on $B_1 \otimes B_2$ [17].

Suppose that $B = B_1 \otimes \dots \otimes B_n$ is a tensor product of KR crystals. For $1 \leq i \leq n-1$, let σ_i be the $A_{\ell-1}^{(1)}$ -crystal isomorphism of B , which acts as $\sigma_{B_i, B_{i+1}}$ on $B_i \otimes B_{i+1}$ and as identity elsewhere, and let H_i be the function on B given by $H_i(b_1 \otimes \dots \otimes b_n) = H_{B_i, B_{i+1}}(b_i \otimes b_{i+1})$. The *energy function* $D_B : B \longrightarrow \mathbb{Z}$ is defined to be

$$(3.6) \quad D_B(b) = \sum_{1 \leq i < j \leq n} H_i(\sigma_{i+1} \sigma_{i+2} \dots \sigma_{j-1}(b)) \quad (b \in B),$$

which plays a very important role in the study of finite affine crystals (cf. [9, 10]). Note that D_B is constant on each connected component in B as an $A_{\ell-1}$ -crystal, which therefore gives a natural q -analogue of the branching multiplicities with respect to $A_{\ell-1} \subset A_{\ell-1}^{(1)}$.

3.5. Crystal skew Howe duality. Let $\mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ be the set of $\ell \times n$ matrices $\mathbf{m} = (m_{ij})$ such that $m_{ij} = 0, 1$ for $1 \leq i \leq \ell$ and $1 \leq j \leq n$.

For $1 \leq i \leq \ell$, let $\mathbf{m}_{(i)}$ denote the i th row of \mathbf{m} . We may identify each $\mathbf{m}_{(i)}$ with an $[n]$ -semistandard tableau of single column whose entries are the column indices

j with $m_{ij} = 1$, and hence regard $\mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ as an A_{n-1} -crystal by identifying \mathbf{m} with $\mathbf{m}_{(\ell)} \otimes \cdots \otimes \mathbf{m}_{(1)}$ with respect to $\tilde{e}_j, \tilde{f}_j, \varepsilon_j, \varphi_j$ for $1 \leq j \leq n-1$.

For $1 \leq j \leq n$, let $\mathbf{m}^{(j)}$ denote the j th column of \mathbf{m} . In the same way, we regard $\mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ as an $A_{\ell-1}$ -crystal by identifying \mathbf{m} with $\mathbf{m}^{(1)} \otimes \cdots \otimes \mathbf{m}^{(n)}$ with respect to $\tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i$ for $1 \leq i \leq \ell-1$.

Then $\mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ is an $(A_{\ell-1}, A_{n-1})$ -bicrystal, that is, $\tilde{x}_i \tilde{x}_j = \tilde{x}_j \tilde{x}_i$ for all i, j , $x = e, f$ and $\mathbf{x} = \mathbf{e}, \mathbf{f}$, and the well-known (dual) RSK correspondence

$$(3.7) \quad \mathbf{M}_{\ell \times n}(\mathbb{Z}_2) \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}} SST_{[\ell]}(\lambda) \times SST_{[n]}(\lambda')$$

is a bicrystal isomorphism [6, 23]. Here λ' denotes the conjugate of λ . This can be viewed as a crystal version of *skew* $(\mathfrak{gl}_{\ell}, \mathfrak{gl}_n)$ -Howe duality (cf. [13]).

Moreover, $\mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ is an $A_{\ell-1}^{(1)}$ -crystal with respect to \tilde{e}_i and \tilde{f}_i for $0 \leq i \leq \ell-1$, since each column of $\mathbf{m} \in \mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ can be considered as an element in a KR crystal $B^{r,1}$ for some $1 \leq r \leq \ell-1$ or a trivial crystal. Note that \tilde{e}_0 and \tilde{f}_0 do not commute with \tilde{e}_j and \tilde{f}_j for $1 \leq j \leq n-1$, in general.

The affine $A_{\ell-1}^{(1)}$ -crystal $\mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ is a union of tensor product of KR-crystals $B_1 \otimes \cdots \otimes B_n$. Thus, we can define the energy function D on $\mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ as in (3.6), where we normalize the local energy function by requiring $H_{B_i, B_{i+1}}(u_i \otimes u_{i+1}) = 0$ for KR crystals B_i with the classical highest weight elements $u_i \in B_i$ for $1 \leq i \leq n$. Then via (3.7) we can rewrite D in terms of statistics on A_{n-1} -crystal as follows.

Proposition 3.1. *For $\mathbf{m} \in \mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$, we have*

$$D(\mathbf{m}) = - \sum_{\alpha \in \Delta_{n-1}^+} \min\{\varepsilon_{\alpha}(\mathbf{m}), \varphi_{\alpha}(\mathbf{m})\}.$$

Proof. We may assume that $\mathbf{m} \in B = B_1 \otimes \cdots \otimes B_n$, where $B_j = SST_{[\ell]}(1^{t_j}) = B^{t_j,1}$ for some t_j ($1 \leq j \leq n$). We can check in a straightforward manner by using the bicrystal structure on $\mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ that for $1 \leq j \leq n-1$,

$$(3.8) \quad \begin{aligned} & \cdot \sigma_{B_j, B_{j+1}} \text{ on } B_j \otimes B_{j+1} \text{ coincides with the } \mathfrak{S}_n\text{-action } S_j \text{ on } \mathbf{m}, \\ & \cdot H_{B_j, B_{j+1}}(\mathbf{m}^{(j)} \otimes \mathbf{m}^{(j+1)}) = -\min\{\varepsilon_j(\mathbf{m}), \varphi_j(\mathbf{m})\}, \end{aligned}$$

(cf. [30, Section 3.5]), where the second statement of (3.8) may be understood as a crystal-theoretic interpretation of [30, Rule 3.10]. Then it follows from (3.5) and (3.8) that

$$H_s(\sigma_{s+1} \sigma_{s+2} \cdots \sigma_{t-1}(\mathbf{m})) = -\min\{\varepsilon_{\alpha}(\mathbf{m}), \varphi_{\alpha}(\mathbf{m})\} \quad (1 \leq s < t \leq n),$$

where $\alpha = \epsilon_s - \epsilon_t$. Hence, we get $D(\mathbf{m}) = -\sum_{\alpha \in \Delta_{n-1}^+} \min\{\varepsilon_{\alpha}(\mathbf{m}), \varphi_{\alpha}(\mathbf{m})\}$. \square

Combining with the result of Nakayashiki and Yamada [30] (see also [31, 32] for its generalisation), we obtain the following intrinsic characterization of charge statistic on a regular A_{n-1} -crystal.

Theorem 3.2. *Let B be a regular A_{n-1} -crystal. For $b \in B$, we have*

$$\text{charge}(b) = \sum_{\alpha \in \Delta_{n-1}^+} \min\{\varepsilon_\alpha(b), \varphi_\alpha(b)\}.$$

Proof. Given $b \in B$, we may assume that $b \in SST_{[n]}(\lambda)$ for some $\lambda \in \mathcal{P}_n$ up to a shift of its weight by $d(\epsilon_1 + \dots + \epsilon_n)$ ($d \geq 0$), say $b = T$. Let \mathbf{m} be the unique matrix in $\mathbf{M}_{\ell \times n}(\mathbb{Z}_2)$ such that $\mathbf{m}_{(i)}$ corresponds to the i th column of T from the left-most column of λ . Since $\text{charge}(b)$ is invariant under the Weyl group action, we may also assume that $\text{wt}(b)$ is dominant, which corresponds to a partition $\mu = (\mu_1, \dots, \mu_n)$. Then $\mathbf{m} \in B^{\mu_1, 1} \otimes \dots \otimes B^{\mu_n, 1}$ as an $A_{\ell-1}^{(1)}$ -crystal. Since we have $c(T) = -D(\mathbf{m})$ by [30, Section 4.1], we have by Proposition 3.1,

$$\text{charge}(b) = c(T) = -D(\mathbf{m}) = \sum_{\alpha \in \Delta_{n-1}^+} \min\{\varepsilon_\alpha(b), \varphi_\alpha(b)\}.$$

This completes the proof. \square

4. A COMBINATORIAL ENERGY FUNCTION

In this section, we introduce a combinatorial energy function D on $\mathcal{F}_{A/B}^\mu$ for $\mu \in \mathbb{Z}_+^n$, which realizes $Q_{A/B}^\mu$ in (2.9) as a graded character of $\mathcal{F}_{A/B}^\mu$.

4.1. Combinatorial energy function. Consider $SST_{A/B}(k_1) \times SST_{A/B}(k_2)$ for $k_1, k_2 \in \mathbb{Z}$. Let $T_j = (T_j^+, T_j^-) \in SST_{A/B}(k_j)$ be given for $j = 1, 2$ with

$$\text{wt}_{A/B}(T_j) = \sum_{a \in A} m_{aj} \epsilon_a - \sum_{b \in B} m_{bj} \epsilon_b.$$

First, we define a *local energy function*

$$\mathbf{H} : SST_{A/B}(k_1) \times SST_{A/B}(k_2) \longrightarrow \mathbb{Z},$$

following the steps below:

- (H-1) Choose finite subsets $A^\circ \subset A$ and $B^\circ \subset B$ such that $T_j \in SST_{A^\circ/B^\circ}(k_j)$ for $j = 1, 2$. To each $i \in A^\circ \sqcup B^\circ$, we assign a sequence of \pm signs as follows:

$$s_i = \begin{cases} \underbrace{- \cdots -}_{m_{i2}} \underbrace{+ \cdots +}_{m_{i1}} & \text{if } i \in \mathcal{A}_0, \\ \underbrace{+}_{m_{i1}} \underbrace{-}_{m_{i2}} & \text{if } i \in \mathcal{A}_1, \\ \underbrace{- \cdots -}_{m_{i1}} \underbrace{+ \cdots +}_{m_{i2}} & \text{if } i \in \mathcal{B}_0, \\ \underbrace{+}_{m_{i2}} \underbrace{-}_{m_{i1}} & \text{if } i \in \mathcal{B}_1. \end{cases}$$

(H-2) Let $\mathbf{s} = s_{T_1, T_2} = (s_{a_k} s_{a_{k-1}} \cdots s_{a_1} s_{b_1} s_{b_2} \cdots s_{b_l})$ be their concatenation where $\mathcal{A}^\circ = \{a_k > \cdots > a_1\}$ and $\mathcal{B}^\circ = \{b_1 < \cdots < b_l\}$, and cancel out all possible $(+ -)$ pairs in \mathbf{s} as far as possible to obtain a reduced sequence

$$\mathbf{s}^{\text{red}} = (\underbrace{- \cdots -}_{\varepsilon} \underbrace{+ \cdots +}_{\varphi}).$$

Then we define

$$H(T_1, T_2) = -\min\{\varepsilon, \varphi\}.$$

Lemma 4.1. *With the same notations as above, we have $\varphi - \varepsilon = k_1 - k_2$.*

Proof. Let p (resp. q) be the total number of $+$'s (resp. $-$'s) in \mathbf{s} . Then

$$p = \sum_{a \in \mathcal{A}} m_{a1} + \sum_{b \in \mathcal{B}} m_{b2}, \quad q = \sum_{a \in \mathcal{A}} m_{a2} + \sum_{b \in \mathcal{B}} m_{b1}.$$

Since $\varphi - \varepsilon = p - q$ and $k_j = \sum_{a \in \mathcal{A}} m_{aj} - \sum_{b \in \mathcal{B}} m_{bj}$ for $j = 1, 2$, we have $\varphi - \varepsilon = k_1 - k_2$. \square

Next, we define a *combinatorial R-matrix*

$$\begin{aligned} \sigma : SST_{\mathcal{A}/\mathcal{B}}(k_1) \times SST_{\mathcal{A}/\mathcal{B}}(k_2) &\longrightarrow SST_{\mathcal{A}/\mathcal{B}}(k_2) \times SST_{\mathcal{A}/\mathcal{B}}(k_1), \\ (T_1, T_2) &\longmapsto (T'_2, T'_1) \end{aligned}$$

where (T'_2, T'_1) is given by moving and rearranging some of the entries in T_1 and T_2 in the following way:

- (σ -1) If $k_1 = k_2$, then put $(T'_2, T'_1) = (T_1, T_2)$.
- (σ -2) If $k_1 > k_2$, then let $y_1, \dots, y_{k_1-k_2}$ be the entries in T_1 or T_2 corresponding to the first $k_1 - k_2$ signs of $+$ in \mathbf{s}^{red} from the left (see Lemma 4.1). For each $1 \leq k \leq k_1 - k_2$, if $y_k \in \mathcal{A}$ (resp. $y_k \in \mathcal{B}$), i.e. $\boxed{y_k}$ appears T_1^+ (resp. T_2^-), then we move it to T_2^+ (resp. T_1^-) and rearrange the entries with respect to the total order on \mathcal{A} (resp. \mathcal{B}).
- (σ -3) If $k_1 < k_2$, then let $x_{k_2-k_1}, \dots, x_1$ be the entries in T_1 or T_2 corresponding to the first $k_2 - k_1$ signs of $-$ in \mathbf{s}^{red} from the right. For each $1 \leq k \leq k_2 - k_1$, if $x_k \in \mathcal{A}$ (resp. $x_k \in \mathcal{B}$), i.e. $\boxed{x_k}$ appears T_2^+ (resp. T_1^-), then we move it

to T_1^+ (resp. T_2^-) and rearrange the entries with respect to the total order on \mathcal{A} (resp. \mathcal{B}).

By definition, it is clear that $\sigma \circ \sigma = \text{id}$.

Example 4.2. (1) If $\mathcal{A} = [\ell]'$ and $\mathcal{B} = \emptyset$, then we have $SST_{\mathcal{A}/\mathcal{B}}(k_j) = B^{k_j,1}$ for $j = 1, 2$, and identify $SST_{\mathcal{A}/\mathcal{B}}(k_1) \times SST_{\mathcal{A}/\mathcal{B}}(k_2)$ with $B^{k_1,1} \otimes B^{k_2,1}$. As we have seen in Section 3.5, we regard $\mathbf{M}_{\ell \times 2}(\mathbb{Z}_2)$ as a union of tensor products $B^{r,1} \otimes B^{s,1}$. Since $\mathbf{M}_{\ell \times 2}(\mathbb{Z}_2)$ is an $(A_{\ell-1}, A_1)$ -bicrystal, we can apply ε_1 and φ_1 to (T_1, T_2) in $SST_{\mathcal{A}/\mathcal{B}}(k_1) \times SST_{\mathcal{A}/\mathcal{B}}(k_2)$ (as an A_1 -crystal). It follows from the definition of \mathbf{H} that

$$\mathbf{H}(T_1, T_2) = -\min\{\varepsilon_1(T_1, T_2), \varphi_1(T_1, T_2)\}.$$

Then we have from (3.8) that

$$\mathbf{H}(T_1, T_2) = H_{B^{k_1,1}, B^{k_2,1}}(T_1 \otimes T_2).$$

Hence \mathbf{H} on $B^{k_1,1} \times B^{k_2,1}$ coincides with the local energy function H on $B^{k_1,1} \otimes B^{k_2,1}$ normalized by $H(u_{k_1,1} \otimes u_{k_2,1}) = 0$.

(2) If $\mathcal{A} = [\ell]$ and $\mathcal{B} = \emptyset$, then we have $SST_{\mathcal{A}/\mathcal{B}}(k_j) = B^{1,k_j}$ for $j = 1, 2$, and identify $SST_{\mathcal{A}/\mathcal{B}}(k_1) \times SST_{\mathcal{A}/\mathcal{B}}(k_2)$ with $B^{1,k_1} \otimes B^{1,k_2}$. As we have seen (1), we consider the crystal version of $(\mathfrak{gl}_n, \mathfrak{gl}_2)$ -Howe duality on the union of tensor products $B^{1,r} \otimes B^{1,s}$. Then, by the definition of \mathbf{H} , one can show that

$$\mathbf{H}(T_1, T_2) = -\min\{\varepsilon_1(T_1, T_2), \varphi_1(T_1, T_2)\},$$

for $(T_1, T_2) \in SST_{\mathcal{A}/\mathcal{B}}(k_1) \times SST_{\mathcal{A}/\mathcal{B}}(k_2)$. Interpreting [30, Rule 3.11] from a point of view of crystal bases theory as (3.8), we can check that the map $-\mathbf{H}$ on $SST_{\mathcal{A}/\mathcal{B}}(k_1) \times SST_{\mathcal{A}/\mathcal{B}}(k_2)$ coincides with the local energy function H on $B^{1,k_2} \otimes B^{1,k_1}$ (in reverse order) normalized by $H(u_{1,k_2} \otimes u_{1,k_1}) = \min\{k_1, k_2\}$ (cf. [30, Section 3]). In both cases (1) and (2), σ is equal to the combinatorial R -matrix σ .

(3) Suppose that \mathcal{A} is finite with $|\mathcal{A}_0| = n$ and $|\mathcal{A}_1| = m$ and $\mathcal{B} = \emptyset$. Then $SST_{\mathcal{A}/\mathcal{B}}(k)$ for $k \geq 1$ can be viewed as a crystal over the quantum superalgebra $U_q(\mathfrak{gl}_{m|n})$ [1]. It would be very nice to find a representation theoretical meaning of \mathbf{H} and σ from finite-dimensional modules over the quantum affine superalgebra $U_q(\widehat{\mathfrak{sl}}_{m|n})$ [34].

(4) Let $\mathbf{T} = (T_1, T_2, T_3)$ be as in Example 2.3. Since

$$\text{wt}(T_2) = \epsilon_{0'} + \epsilon_{2'} + \epsilon_{6'} + \epsilon_{7'} - \epsilon_{4'} - \epsilon_{2'} - \epsilon_{-1'},$$

$$\text{wt}(T_3) = \epsilon_{4'} + \epsilon_{5'} - \epsilon_{-2'} - \epsilon_{-1'},$$

we have $\mathbf{s}_{T_2, T_3} = (+ + - - + + - + - + -)$. Thus, the reduced sequence $\mathbf{s}^{\text{red}} = (+ +)$ gives $\mathbf{H}(T_2, T_3) = 0$ and

$$\sigma(T_2, T_3) = (T_2', T_3') \in SST_{\mathcal{A}/\mathcal{B}}(0) \times SST_{\mathcal{A}/\mathcal{B}}(1),$$

where

$$T'_2 = (\begin{bmatrix} 0' & 6' & 7' \end{bmatrix}, \begin{bmatrix} -4' & -2' & -1' \end{bmatrix}), \quad T'_3 = (\begin{bmatrix} 2' & 4' & 5' \end{bmatrix}, \begin{bmatrix} -2' & -1' \end{bmatrix}).$$

In the same manner, we compute

$$\begin{aligned} s_{T_1, T_2} &= (- \ - \ + \ + \ + \ - \ + \ + \ - \ + \ - \ - \ + \ +), \\ s_{T_1, T'_2} &= (- \ - \ + \ + \ + \ + \ + \ - \ + \ - \ - \ + \ +), \end{aligned}$$

which yield $H(T_1, T_2) = -2$ and $H(T_1, T'_2) = -2$.

While $SST_{\mathcal{A}/\mathcal{B}}(k)$ for $k \in \mathbb{Z}$ produces a character of a level one integrable highest weight module over $U_q(\mathfrak{gl}_\infty)$, it also corresponds to a KR module over $U_q(\widehat{\mathfrak{sl}}_\infty)$, where $\widehat{\mathfrak{sl}}_\infty$ is an affinization of \mathfrak{sl}_∞ [11]. As in (3), we expect that H and σ are closely related with the theory of KR modules over $U_q(\widehat{\mathfrak{sl}}_\infty)$.

Now, we fix $n \geq 1$. For simplicity, we put

$$\begin{aligned} (4.1) \quad \mathcal{F}^n &= \mathcal{F}_{\mathcal{A}/\mathcal{B}}^n, \\ \mathcal{F}^\mu &= SST_{\mathcal{A}/\mathcal{B}}(\mu_1) \times \cdots \times SST_{\mathcal{A}/\mathcal{B}}(\mu_n), \end{aligned}$$

for $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$. Clearly, we have $\mathcal{F}^n = \bigsqcup_{\mu \in \mathbb{Z}^n} \mathcal{F}^\mu$. For $1 \leq i \leq n-1$, let σ_i be the map on $\mathcal{F}^\mu \subset \mathcal{F}^n$, which acts as σ on $SST_{\mathcal{A}/\mathcal{B}}(\mu_i) \times SST_{\mathcal{A}/\mathcal{B}}(\mu_{i+1})$ and as identity elsewhere, and let H_i be the map on \mathcal{F}^n given by $H_i(T_1, \dots, T_n) = H(T_i, T_{i+1})$ for $(T_1, \dots, T_n) \in \mathcal{F}^n$.

We define a *combinatorial energy function* $D : \mathcal{F}^n \rightarrow \mathbb{Z}$ by

$$D(\mathbf{T}) = \sum_{1 \leq i < j \leq n} H_i(\sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1}(\mathbf{T})) \quad (\mathbf{T} \in \mathcal{F}^n).$$

Then we have the following, which is a generalization of [30]. The proof is given in the next section.

Theorem 4.3. *For $\mathbf{T} \in \mathcal{F}^n$, we have*

$$D(\mathbf{T}) = -\text{charge}(Q_{\mathbf{T}}),$$

where $Q_{\mathbf{T}}$ is the rational semistandard tableau corresponding to \mathbf{T} under the RSK map $\kappa_{\mathcal{A}/\mathcal{B}}$ on \mathcal{F}^n in Theorem 2.4. In particular, we have $D(\mathbf{T}) = D(\mathbf{T}')$ for $\mathbf{T}, \mathbf{T}' \in \mathcal{F}^n$ such that $Q_{\mathbf{T}} = Q_{\mathbf{T}'}$.

Example 4.4. Continuing Example 4.2 (4), we have

$$D(\mathbf{T}) = H(T_1, T_2) + H(T_1, T'_2) + H(T_2, T_3) = -4.$$

Since

$$\text{charge}(Q_{\mathbf{T}}) = c \left(\begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & & & & \\ \hline \end{array} \right) = 4,$$

we have $D(\mathbf{T}) = -\text{charge}(Q_{\mathbf{T}})$.

As a consequence, we obtain a combinatorial realization of (2.9) in terms of \mathbf{D} .

Theorem 4.5. *For $\mu \in \mathbb{Z}_+^n$, we have*

$$Q_\mu^{A/\mathcal{B}} = \sum_{\mathbf{T} \in \mathcal{F}^\mu} q^{-\mathbf{D}(\mathbf{T})} \mathbf{x}_{A/\mathcal{B}}^{\mathbf{T}},$$

where $\mathbf{x}_{A/\mathcal{B}}^{\mathbf{T}} = \prod_{i=1}^n \mathbf{x}_{A/\mathcal{B}}^{T_i}$ for $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{F}^\mu$.

Proof. Restricting $\kappa_{A/\mathcal{B}}$ to \mathcal{F}^μ , we have a weight preserving bijection:

$$\kappa_{A/\mathcal{B}} : \mathcal{F}^\mu \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_{A/\mathcal{B},n}} SST_{A/\mathcal{B}}(\lambda) \times SST_{[n]}(\lambda)_\mu.$$

Thus, the assertion follows from (2.9), (3.2), and Theorem 4.3. \square

4.2. Proof of Theorem 4.3. The proof is given in two steps. We will first define a regular A_{n-1} -crystal structure on \mathcal{F}^n and show that $-\mathbf{D}$ is equal to the charge on \mathcal{F}^n as an A_{n-1} -crystal. Next, we will show that the RSK type correspondence $\kappa_{A/\mathcal{B}}$ for parabolically semistandard tableaux is an A_{n-1} -crystal isomorphism, which is a key part in the proof of Theorem 4.3.

Let us define an A_{n-1} -crystal structure on \mathcal{F}^n . Let $\mathbf{M}_{A/\mathcal{B},n}$ be the set of matrices $\mathbf{m} = (m_{ij})$ with non-negative integral entries ($i \in \mathcal{A} \sqcup \mathcal{B}$, $j \in [n]$) satisfying (1) $\sum_{i,j} m_{ij} < \infty$, (2) $m_{ij} \in \{0, 1\}$ if i is odd. Note that for $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{F}^n$ with

$$\text{wt}_{A/\mathcal{B}}(T_j) = \sum_{a \in \mathcal{A}} m_{aj} \epsilon_a - \sum_{b \in \mathcal{B}} m_{bj} \epsilon_b,$$

for $1 \leq j \leq n$, the map sending \mathbf{T} to $\mathbf{m} = (m_{ij})$ gives a natural bijection from \mathcal{F}^n to $\mathbf{M}_{A/\mathcal{B},n}$.

Let $\mathbf{m} \in \mathbf{M}_{A/\mathcal{B},n}$ be given. For $i \in \mathcal{A} \sqcup \mathcal{B}$, let $\mathbf{m}_{(i)} = (m_{ij})_{j \in [n]}$ be the i th row of \mathbf{m} , and set $|\mathbf{m}_{(i)}| = \sum_{j \in [n]} m_{ij}$. Let $\lambda^{(i)} \in \mathbb{Z}_+^n$ be given by

$$\lambda^{(i)} = \begin{cases} (|\mathbf{m}_{(i)}|, 0, \dots, 0), & \text{if } i \in \mathcal{A}_0, \\ (1^{|\mathbf{m}_{(i)}|}, 0, \dots, 0), & \text{if } i \in \mathcal{A}_1, \\ (0, \dots, 0, -|\mathbf{m}_{(i)}|), & \text{if } i \in \mathcal{B}_0, \\ (0, \dots, 0, -1^{|\mathbf{m}_{(i)}|}), & \text{if } i \in \mathcal{B}_1. \end{cases}$$

We identify $\mathbf{m}_{(i)}$ with a unique rational semistandard tableau $T^{(i)} \in SST_{[n]}(\lambda^{(i)})$ such that

$$\text{wt}_{[n]}(T^{(i)}) = \begin{cases} \sum_{j \in [n]} m_{ij} \epsilon_j, & \text{if } i \in \mathcal{A}, \\ -\sum_{j \in [n]} m_{ij} \epsilon_j, & \text{if } i \in \mathcal{B}. \end{cases}$$

For example, if $a \in \mathcal{A}_0$ and $b \in \mathcal{B}_1$, then we have

$$\begin{aligned} \mathbf{m}_{(a)} = (2, 0, 1, 2) &\longleftrightarrow T^{(a)} = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 4 \\ \hline \end{array} \in SST_{[4]}(5, 0, 0, 0), \\ \mathbf{m}_{(b)} = (1, 1, 0, 1) &\longleftrightarrow T^{(b)} = \begin{array}{|c|} \hline -4 \\ \hline -2 \\ \hline -1 \\ \hline \end{array} \in SST_{[4]}(0, -1, -1, -1). \end{aligned}$$

Then we define a regular A_{n-1} -crystal structure on $\mathbf{M}_{\mathcal{A}/\mathcal{B}, n}$ and hence on \mathcal{F}^n via the correspondence

$$(4.2) \quad \mathbf{m} \longleftrightarrow \bigotimes_{a \in \mathcal{A}}^{\leftarrow} T^{(a)} \otimes \bigotimes_{b \in \mathcal{B}}^{\rightarrow} T^{(b)}.$$

Here we understand $\bigotimes_{a \in \mathcal{A}}^{\leftarrow} T^{(a)}$ as a tensor product with respect to the reverse total order on \mathcal{A} . Since $T^{(a)}$ is an empty tableau except for finitely many $a \in \mathcal{A}$, it is well-defined. Similarly, $\bigotimes_{b \in \mathcal{B}}^{\rightarrow} T^{(b)}$ is a tensor product with respect to the total order on \mathcal{B} . One may assume that the row indices of $\mathbf{m} \in \mathbf{M}_{\mathcal{A}/\mathcal{B}, n}$ are parametrized by $\mathcal{B}^\pi * \mathcal{A}$, and we read each row in \mathbf{m} from bottom to top.

Since \mathcal{F}^n is an A_{n-1} -crystal, one can consider $\varepsilon_\alpha(\mathbf{T})$ and $\varphi_\alpha(\mathbf{T})$ for $\mathbf{T} \in \mathcal{F}^n$ and $\alpha \in \Delta_{n-1}^+$ as in (3.5). By definitions of H and σ , we can check that

$$(4.3) \quad H_i(\mathbf{T}) = -\min\{\varepsilon_i(\mathbf{T}), \varphi_i(\mathbf{T})\}, \quad \sigma_i(\mathbf{T}) = S_i(\mathbf{T}),$$

for $1 \leq i \leq n-1$. In particular σ_i 's satisfy the braid relations. Thus, combining (4.3) with $\varepsilon_\alpha(\mathbf{T})$ and $\varphi_\alpha(\mathbf{T})$, we obtain the following, which generalizes Proposition 3.1.

Proposition 4.6. *For $\mathbf{T} \in \mathcal{F}^n$, we have*

$$D(\mathbf{T}) = - \sum_{\alpha \in \Delta_{n-1}^+} \min\{\varepsilon_\alpha(\mathbf{T}), \varphi_\alpha(\mathbf{T})\}.$$

Moreover, the regular A_{n-1} -crystal structure of \mathcal{F}^n enables us to consider the charge of $\mathbf{T} \in \mathcal{F}^n$. By Theorem 3.2, we have

Corollary 4.7. *For $\mathbf{T} \in \mathcal{F}^n$, we have $D(\mathbf{T}) = -\text{charge}(\mathbf{T})$.*

This also immediately implies that $D \circ \sigma_i = D$ for $1 \leq i \leq n-1$.

Next, we interpret the map $\kappa_{\mathcal{A}/\mathcal{B}}$

$$\kappa_{\mathcal{A}/\mathcal{B}} : \mathcal{F}^n \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B}, n}} SST_{\mathcal{A}/\mathcal{B}}(\lambda) \times SST_{[n]}(\lambda).$$

from a viewpoint of crystal. We assume that the right-hand side is an A_{n-1} -crystal, where the operators \tilde{e}_j and \tilde{f}_j act on the second factor $SST_{[n]}(\lambda)$.

Theorem 4.8. *The map $\kappa_{\mathcal{A}/\mathcal{B}}$ is an A_{n-1} -crystal isomorphism.*

Proof. Let us recall the bijections ϱ_{col} and ϱ_{row} given in (2.1). Suppose $\mathcal{B} = \emptyset$ and write $\mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}/\emptyset}$. Extending ϱ_{col} and ϱ_{row} to $\mathcal{F}_{\mathcal{A}}^n$, we have bijections

$$\varrho_{\text{col}}, \varrho_{\text{row}} : \mathcal{F}_{\mathcal{A}}^n \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_{\mathcal{A}} \cap \mathcal{P}_n} SST_{\mathcal{A}}(\lambda) \times SST_{[n]}(\lambda),$$

which are indeed the RSK correspondences since $\mathcal{F}_{\mathcal{A}}^n$ can be identified with $\mathbf{M}_{\mathcal{A}/\emptyset, n}$. Let $\mathbf{T} \in \mathcal{F}_{\mathcal{A}}^n$ be given with the corresponding matrix $\mathbf{m} \in \mathbf{M}_{\mathcal{A}/\emptyset, n}$. If we write $\varrho_{\text{col}}(\mathbf{T}) = (P_c, Q_c)$ and $\varrho_{\text{row}}(\mathbf{T}) = (P_r, Q_r)$, then

$$(4.4) \quad \bigotimes_{a \in \mathcal{A}}^{\rightarrow} \mathbf{m}_a \equiv Q_c, \quad \bigotimes_{a \in \mathcal{A}}^{\leftarrow} \mathbf{m}_a \equiv Q_r.$$

Indeed, the first equivalence follows from [23, Theorem 3.11] on $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$ -bicrystal isomorphism over general linear Lie superalgebras, where we replace $\mathfrak{gl}_{u|v}$ with $\mathfrak{gl}_{n|0}$ and $\mathfrak{gl}_{m|n}$ with a Lie superalgebra $\mathfrak{gl}_{\mathcal{A}}$ associated to \mathcal{A} (see also [25, Lemma 4.9]). Similarly, the second equivalence can be obtained by changing the parity of \mathcal{A} and applying [23, Theorem 4.5].

Let $\mathbf{T} \in \mathcal{F}^n$ be given, and \mathbf{m} the corresponding matrix in $\mathbf{M}_{\mathcal{A}/\mathcal{B}, n}$. Let $\kappa_{\mathcal{A}/\mathcal{B}}(\mathbf{T}) = (P_{\mathbf{T}}, Q_{\mathbf{T}})$. We keep the same notations Q , Q^{\vee} , and U_R in $(\kappa-1)$ -($\kappa-4$) in Section 2.3. It follows from (3.1), $(\kappa-1)$, $(\kappa-2)$, and the first equivalence in (4.4) that

$$Q^{\vee} \equiv \left(\bigotimes_{b \in \mathcal{B}^{\pi}}^{\rightarrow} \mathbf{m}_b^{\vee} \right)^{\vee} \equiv \left(\bigotimes_{b \in \mathcal{B}}^{\leftarrow} \mathbf{m}_b^{\vee} \right)^{\vee} \equiv \bigotimes_{b \in \mathcal{B}}^{\rightarrow} \mathbf{m}_b.$$

We should remark that on $\mathbf{M}_{\emptyset/\mathcal{B}, n}$ the A_{n-1} -crystal structure is dual to that of $\mathbf{M}_{\mathcal{A}/\emptyset, n}$. Moreover, since $k < a$ in $[n] * \mathcal{A}$ for all $k \in [n]$ and $a \in \mathcal{A}$, (3.1) and the second equivalence in (4.4) give

$$Q_{\mathbf{T}} \equiv U_R \equiv \left(\bigotimes_{a \in \mathcal{A}}^{\leftarrow} \mathbf{m}_a \right) \otimes Q^{\vee} \equiv \left(\bigotimes_{a \in \mathcal{A}}^{\leftarrow} \mathbf{m}_a \right) \otimes \left(\bigotimes_{b \in \mathcal{B}}^{\rightarrow} \mathbf{m}_b \right) \equiv \mathbf{T},$$

which implies that $\kappa_{\mathcal{A}/\mathcal{B}}$ is a morphism of A_{n-1} -crystals. Thus the assertion follows from Theorem 2.4. \square

Corollary 4.9. *For $\mathbf{T} \in \mathcal{F}^n$, $\text{charge}(\mathbf{T}) = \text{charge}(Q_{\mathbf{T}})$.*

Now, Theorem 4.3 follows from Corollaries 4.7 and 4.9. This completes the proof.

Example 4.10. Continuing Example 2.3, we have the matrix \mathbf{m} corresponding to \mathbf{T} as follows:

$$\mathbf{m} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline -4' & \bullet & \bullet & \cdot \\ -3' & \bullet & \cdot & \cdot \\ -2' & \cdot & \bullet & \bullet \\ -1' & \cdot & \bullet & \bullet \\ 0' & \bullet & \bullet & \cdot \\ 1' & \bullet & \cdot & \cdot \\ 2' & \cdot & \bullet & \cdot \\ 3' & \bullet & \cdot & \cdot \\ 4' & \bullet & \cdot & \bullet \\ 5' & \bullet & \cdot & \bullet \\ 6' & \cdot & \bullet & \cdot \\ 7' & \cdot & \bullet & \cdot \end{array}$$

where \cdot and \bullet denote 0 and 1 respectively. This yields

$$\begin{aligned} T^{(0')} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T^{(1')} = T^{(3')} = \begin{bmatrix} 1 \end{bmatrix}, \quad T^{(2')} = T^{(6')} = T^{(7')} = \begin{bmatrix} 2 \end{bmatrix}, \quad T^{(4')} = T^{(5')} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \\ T^{(-1')} &= T^{(-2')} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}, \quad T^{(-3')} = \begin{bmatrix} -1 \end{bmatrix}, \quad T^{(-4')} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}. \end{aligned}$$

Thus, using the Schensted's bumping algorithm, we see

$$\begin{aligned} T^{(7')} \otimes T^{(6')} \otimes T^{(5')} \otimes T^{(4')} \otimes T^{(3')} \otimes T^{(2')} \otimes T^{(1')} \otimes T^{(0')} &\equiv \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 & & & \\ \hline \end{array}, \\ T^{(-4')} \otimes T^{(-3')} \otimes T^{(-2')} \otimes T^{(-1')} &\equiv \begin{bmatrix} 3 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \end{bmatrix} \equiv \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 3 & & & \\ \hline \end{array} \equiv Q^\vee. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbf{T} &\equiv \bigotimes_{a \in \mathbb{Z}'_{\geq 0}}^{\leftarrow} T^{(a)} \otimes \bigotimes_{b \in \mathbb{Z}'_{< 0}}^{\rightarrow} T^{(b)} \\ &\equiv \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 & & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 3 & & & \\ \hline \end{array} \\ &\equiv Q_{\mathbf{T}}. \end{aligned}$$

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